

# ON THE PROBABILITY DISTRIBUTION OF THE GCD AND LCM OF $r$ -TUPLES OF INTEGERS

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**ABSTRACT.** This paper is devoted to the study of statistical properties of the greatest common divisor and the least common multiple of random samples of positive integers.

## 1. INTRODUCTION

For any given integer  $n \geq 2$ , let us denote by  $X_1^{(n)}, X_2^{(n)}, \dots$  a sequence of independent random variables uniformly distributed in  $\{1, 2, \dots, n\}$  and defined in a certain given probability space endowed with a probability  $\mathbf{P}$ . For a concrete realization we may take the unit interval with Lebesgue measure and Borel  $\sigma$ -algebra as the probability space, and for  $j \geq 1$ , the variable  $X_j^{(n)}$  whose value at  $\omega \in [0, 1]$  is 1 plus the  $j$ -th digit of the expansion in base  $n$  of  $\omega$ .

We are interested in studying the probability distributions of the random variables

$$\gcd(X_1^{(n)}, \dots, X_r^{(n)}) \quad \text{and} \quad \text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) \quad \text{for } r \geq 2,$$

*i.e.*, the gcd and the lcm of random  $r$ -tuples of integers.

Dirichlet's basic and classical result asserts that the proportion of coprime pairs of integers in  $\{1, 2, \dots, n\}$ ,

$$\frac{1}{n^2} \# \{(i, j) : 1 \leq i, j \leq n; \gcd(i, j) = 1\},$$

tends to  $1/\zeta(2) = 6/\pi^2$  as  $n$  tends to  $\infty$ , which, in the probabilistic setting introduced above, reads:

$$\lim_{n \rightarrow \infty} \mathbf{P}(\gcd(X_1^{(n)}, X_2^{(n)}) = 1) = \frac{1}{\zeta(2)}.$$

See, for instance, [14], Theorem 332.

The limiting behavior of the whole distribution of the gcd of pairs follows immediately from the above: for each integer  $k \geq 1$ ,

$$\lim_{n \rightarrow \infty} \mathbf{P}(\gcd(X_1^{(n)}, X_2^{(n)}) = k) = \frac{1}{\zeta(2)} \frac{1}{k^2}.$$

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The probability distributions and moments of the gcd and the lcm of pairs of integers are described asymptotically in the following two known theorems:

**Theorem A.** a) *The mass function of the gcd of pairs of integers satisfies*

$$(1.1) \quad \mathbf{P}(\gcd(X_1^{(n)}, X_2^{(n)}) = k) = \frac{1}{\zeta(2)} \frac{1}{k^2} + O\left(\frac{1 + \ln(n/k)}{nk}\right) \quad \text{for } 1 \leq k \leq n.$$

b) *The moments of the gcd of pairs of integers are given by*

$$(1.2) \quad \mathbf{E}(\gcd(X_1^{(n)}, X_2^{(n)})) = \frac{1}{\zeta(2)} \ln(n) + C + O\left(\frac{\ln(n)}{\sqrt{n}}\right);$$

$$(1.3) \quad \mathbf{E}(\gcd(X_1^{(n)}, X_2^{(n)})^q) = \frac{n^{q-1}}{(q+1)} \left[ \frac{2\zeta(q)}{\zeta(q+1)} - 1 \right] + O(n^{q-2} \ln(n)), \quad \text{for } q \geq 2;$$

The estimate (1.1) follows directly from the usual bounds in Dirichlet's result. Estimates (1.2) and (1.3) appear, for instance, in a paper of Cohen (take  $g(n) = 1$  if  $n = 1$ , and  $g(n) = 0$  elsewhere, in the notation of Theorem in page 168 of [8]). The constant  $C$  is recorded explicitly there. See also [9].

**Theorem B.** a) *For  $0 < t \leq 1$ , the distribution function of the lcm of pairs of integers satisfies:*

$$(1.4) \quad \mathbf{P}(\text{lcm}(X_1^{(n)}, X_2^{(n)}) \leq tn^2) = 1 - \frac{1}{\zeta(2)} \sum_{j=1}^{\lfloor 1/t \rfloor} \frac{1 - jt(1 - \ln(jt))}{j^2} + O_t\left(\frac{\ln(n)}{n}\right).$$

b) *The moments of the lcm of pairs of integers satisfy*

$$(1.5) \quad \mathbf{E}(\text{lcm}(X_1^{(n)}, X_2^{(n)})^q) = \frac{\zeta(q+2)}{\zeta(2)(q+1)^2} n^{2q} + O(n^{2q-1} \ln(n)), \quad \text{for } q \geq 1;$$

Observe that the bound of the error term in (1.4) depends on  $t$ . (Throughout the paper, the notation  $O_t$  means that the constant in the  $O$ -bound depends only on  $t$ .)

The estimate (1.4) is more involved than the corresponding result for the gcd; it is due to Diaconis and Erdős [9]. Notice that the denominator  $j^2$  in formula (1.4) is missing in the statement of Theorem 1 in [9].

Result (1.5) can be traced back all the way to Cesàro (see [5], page 248). See also Theorem 2 in [9].

In this note we will prove a number of asymptotic results (Theorem A' and Theorems 1–3) concerning the probability distributions (mass distribution and moments) of the gcd and the lcm of  $r$ -tuples of integers, for  $r \geq 3$ . The case of gcd is quite direct, but not so the case of lcm, as we see later.

Theorem A can be readily extended to higher moments. It is worth recording it, as we shall use these estimates elsewhere (see [11]).

**Theorem A'.** *Let  $r \geq 3$ .*

a) *For  $1 \leq k \leq n$ ,*

$$(1.6) \quad \mathbf{P}(\gcd(X_1^{(n)}, \dots, X_r^{(n)}) = k) = \frac{1}{k^r \zeta(r)} + O\left(\frac{1}{n k^{r-1}}\right).$$

b) *Let  $q$  be a positive integer.*

b1) If  $1 \leq q \leq r - 2$ ,

$$(1.7) \quad \mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^q) = \frac{\zeta(r-q)}{\zeta(r)} + O_r\left(\frac{\ln(n)}{n}\right).$$

b2) If  $q = r - 1$ ,

$$(1.8) \quad \mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^{r-1}) = \frac{\ln(n)}{\zeta(r)} + O_r(1).$$

b3) Finally, for  $q \geq r$ ,

$$(1.9) \quad \mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^q) = D_{r,q} n^{q-r+1} + O_{r,q}(n^{q-r} \ln(n)),$$

where the constant  $D_{r,q}$  is given by

$$D_{r,q} = \frac{1}{(q+1)\zeta(q+1)} \sum_{k=1}^r \binom{r}{k} (-1)^{k+1} \zeta(q-r+k+1).$$

The estimate (1.6) is straightforward, and it appears, with no bound on the error term, in Cesàro ([6], page 293). See also, for instance, [7], [16] and [18]. For the sake of completeness, we prove Theorem A' in Section 3, particularly of (1.9).

Observe that b1) implies in particular that the mean of the gcd of an  $r$ -tuple of integers in  $\{1, 2, \dots, n\}$  has a finite limit for  $r \geq 3$ :

$$\lim_{n \rightarrow \infty} \mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})) = \frac{\zeta(r-1)}{\zeta(r)},$$

reflecting the fact that, on average, the gcd of an  $r$ -tuple is quite close to 1, for moderately large  $r$ . Notice also that the constant  $D_{2,q}$  reduces to

$$\frac{1}{(q+1)} \frac{1}{\zeta(q+1)} (2\zeta(q) - \zeta(q+1)).$$

as in (1.3). We should mention that the asymptotic estimates of moments of gcd above are valid also for non-integer  $q$ , but we limit ourselves to the integer case.

The random behavior of the least common multiple of  $r$ -tuples is subtler for  $r \geq 3$  than for  $r = 2$ . The random variable  $\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})$  could be normalized in several manners; for instance, in terms of  $n^r$ , of its maximum possible value  $\text{lcm}(1, \dots, n)$ , or in terms of the product of the  $X_j$ 's:

$$\frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{n^r}, \quad \frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{\text{lcm}(1, \dots, n)}, \quad \text{or} \quad \frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{X_1^{(n)} \dots X_r^{(n)}}.$$

In all three alternatives, we obtain a random variable with values in  $[0, 1]$ . We will focus on the first alternative (but see Propositions 4.8 and 4.9 for the third one). Recall that Theorem B claims that the sequence of variables

$$\mathcal{L}_n = \frac{\text{lcm}(X_1^{(n)}, X_2^{(n)})}{n^2}, \quad n \geq 1$$

converges in distribution to a random variable  $\mathcal{L}$  with values in  $[0, 1]$  whose complementary distribution function is given by

$$\mathbf{P}(\mathcal{L} > t) = \frac{1}{\zeta(2)} \sum_{j=1}^{\lfloor 1/t \rfloor} \frac{1 - jt(1 - \ln(jt))}{j^2},$$

and whose moments are given by  $\mathbf{E}(\mathcal{L}^q) = \frac{\zeta(q+2)}{\zeta(2)(q+1)^2}$ ,  $q \geq 1$ .

To state our results about the distribution function of the lcm, we introduce the following notation: for  $r \geq 2$  and  $s > 0$ , denote by

$$(1.10) \quad \mathcal{A}_r(s) := \{(x_1, \dots, x_r) : 0 \leq x_1, \dots, x_r \leq 1, x_1 \cdots x_r \leq s\}$$

the part of the unit positive  $r$ -cube where  $x_1 \cdots x_r \leq s$ . We write  $\Omega_r(s)$  for the volume of  $\mathcal{A}_r(s)$ . Observe that  $\Omega_r(s) = 1$  for  $s \geq 1$ . For  $s < 1$ ,

$$(1.11) \quad \Omega_r(s) = s \sum_{k=0}^{r-1} \frac{\ln(1/s)^k}{k!}$$

(see Lemma 2.2). For each  $r \geq 2$ , write

$$(1.12) \quad T_r := \prod_p \left(1 - \frac{1}{p}\right)^{r-1} \left(1 + \frac{r-1}{p}\right).$$

The constant  $T_r$  is the asymptotic proportion of  $r$ -tuples of integers that are pairwise coprime (see [4] and [19]). Clearly,  $\lim_{r \rightarrow \infty} T_r = 0$ ; in fact, it does so very rapidly:  $\lim_{r \rightarrow \infty} T_r^{1/r} \ln(r) = e^{-\gamma}$ , where  $\gamma$  is Euler's constant, see [15]. The first values are:  $T_2 = 1/\zeta(2) \approx 0.60793$ ,  $T_3 \approx 0.28675$ ,  $T_4 \approx 0.11488$ , etc.

**Theorem 1.** *Let  $r \geq 3$ . Then, for  $0 < t \leq 1$ ,*

$$(1.13) \quad \liminf_{n \rightarrow \infty} \mathbf{P}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) \leq tn^r) \geq 1 - \frac{1}{\zeta(r)} \sum_{j=1}^{\infty} \frac{1 - \Omega_r(tj^{r-1})}{j^r},$$

$$(1.14) \quad \limsup_{n \rightarrow \infty} \mathbf{P}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) \leq tn^r) \leq 1 - T_r \sum_{j=1}^{\infty} \frac{1 - \Omega_r(tj^{r-1})}{j^r}.$$

As  $\Omega_r(s) = 1$  for  $s \geq 1$ , the series in (1.13) and (1.14), for each  $0 \leq t \leq 1$ , are actually finite sums (the range extends to those  $j$  such that  $j^{r-1} \leq 1/t$ ). Notice also that the right hand side of (1.13) is a distribution function, with value 0 as  $t \rightarrow 0$ , and value 1 as  $t \rightarrow 1$ . The right hand side of (1.14) is not a distribution function, as it takes the value  $1 - T_r \zeta(r)$  as  $t \rightarrow 0$  (see Figure 1 for a depiction of the case  $r = 3$ ).

Setting  $r = 2$  in Theorem 1, we recover (1.4), since  $T_2 = 1/\zeta(2)$  and  $\Omega_2(s) = s(1 - \ln(s))$ .

For the moments of the lcm of  $r$ -tuples, we prove:

**Theorem 2.** *Let  $r \geq 3$ . For each integer  $q \geq 1$ ,*

$$(1.15) \quad \limsup_{n \rightarrow \infty} \frac{\mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})^q)}{n^{rq}} \leq \frac{1}{\zeta(r)} \frac{\zeta(r(q+1) - q)}{(q+1)^r}.$$

$$(1.16) \quad \liminf_{n \rightarrow \infty} \frac{\mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})^q)}{n^{rq}} \geq T_r \frac{\zeta(r(q+1) - q)}{(q+1)^r}.$$

Again, for  $r = 2$ , since  $T_2 = 1/\zeta(2)$ , we recover (1.5) of Theorem B.

Theorems 1 and 2 do tell us that

$$\mathbf{P}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) > tn^r) \asymp \sum_{j=1}^{\infty} \frac{1 - \Omega_r(tj^{r-1})}{j^r},$$

and

$$\mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})^q) \asymp n^{rq},$$

but, asymptotically, they only provide upper and lower estimates for the distribution function and moments of the lcm of  $r$ -tuples of integers in the limit  $n \rightarrow \infty$ . We have not been able to handle the combinatorics that would lead to establish precise asymptotic estimates for general  $r$ ; but we have studied in detail the case  $r = 3$  and obtained the precise result contained in Theorem 3. To state it, we need to introduce some more notation. We will denote by  $\omega(m)$  the number of *distinct* prime factors of  $m$ , and for each  $r \geq 2$  we will write  $\Upsilon_r(m)$  for the arithmetic function given by  $\Upsilon_r(1) = 1$  and

$$(1.17) \quad \Upsilon_r(m) = \prod_{p|m} \frac{(1 + (r-2)/p)}{(1 + (r-1)/p)} \quad \text{for } m \geq 2.$$

The function  $\Upsilon_r$  is multiplicative, and  $\Upsilon_r(m) < 1$ , for  $m > 1$ . The case  $r = 3$ ,

$$\Upsilon_3(m) = \prod_{p|m} \frac{1 + 1/p}{1 + 2/p},$$

will be of special interest. Finally, we shall denote by  $J$  the Dirichlet series:

$$(1.18) \quad J(s) = \sum_{m=1}^{\infty} \frac{\Upsilon_3(m) 3^{\omega(m)}}{m^s}, \quad \Re(s) > 1.$$

**Theorem 3.** a) For  $0 \leq t \leq 1$ ,

$$(1.19) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbf{P}(\text{lcm}(X_1^{(n)}, X_2^{(n)}, X_3^{(n)}) \leq tn^3) \\ = 1 - T_3 \sum_{j=1}^{\infty} \frac{1}{j^3} \sum_{m=1}^{\infty} \frac{\Upsilon_3(m) 3^{\omega(m)}}{m^2} (1 - \Omega_3(tj^2m)), \end{aligned}$$

b) For integer  $q \geq 1$ ,

$$(1.20) \quad \lim_{n \rightarrow \infty} \frac{\mathbf{E}(\text{lcm}(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})^q)}{n^{3q}} = T_3 \frac{\zeta(2q+3)}{(q+1)^3} J(q+2),$$

Incidentally, observe that the case  $t = 0$  of (1.19) implies the identity

$$(1.21) \quad J(2) = \sum_{m=1}^{\infty} \frac{\Upsilon_3(m) 3^{\omega(m)}}{m^2} = \frac{1}{T_3 \zeta(3)}$$

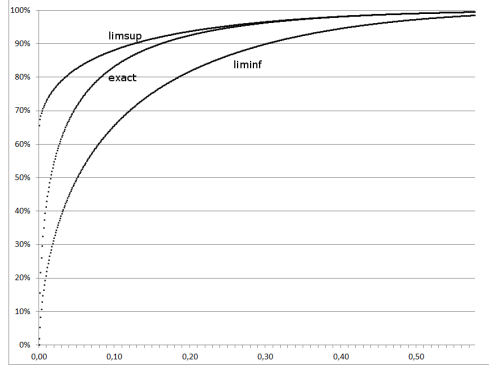
(see Remark 5.2). Figure 1 compares the exact value (1.19) with the lower and upper bounds given in (1.13) and (1.14) for  $r = 3$ .

Finally, we discuss certain *waiting time* questions concerning sequences of successive gcds and lcms

$$\begin{aligned} z_1 = x_1, \quad z_2 = \gcd(x_1, x_2), \quad z_3 = \gcd(x_1, x_2, x_3) \dots, \\ w_1 = x_1, \quad w_2 = \text{lcm}(x_1, x_2), \quad w_3 = \text{lcm}(x_1, x_2, x_3) \dots, \end{aligned}$$

of integers  $x_1, x_2, \dots$  drawn uniformly and independently from  $\{1, \dots, n\}$ .

The sequence  $(z_j)$  decreases almost surely to 1, while the sequence  $(w_j)$  increases almost surely to the number  $\text{lcm}(1, \dots, n)$ . Their respective *expected* waiting times are dealt with in Section 6.

FIGURE 1. Comparison for the case  $r = 3$ .

The paper is organized as follows: we will analyze the properties of the gcd of  $r$ -tuples in Section 3. Section 4 contains the proofs of the results concerning the lcm of  $r$ -tuples. The particular case  $r = 3$  for lcm is studied in Section 5. Finally, Section 6 contains the analysis of those waiting times related to gcd and lcm.

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## 2. PRELIMINARIES AND NOTATION

Throughout the paper, for a vector  $\mathbf{x} = (x_1, \dots, x_r)$  of positive integers, we abbreviate  $\gcd(\mathbf{x}) = \gcd(x_1, \dots, x_r)$  and  $\text{lcm}(\mathbf{x}) = \text{lcm}(x_1, \dots, x_r)$ . Also if  $\beta = (\beta_1, \dots, \beta_r)$  is a vector of positive real numbers we economize and write  $\mathbf{x} \leq \beta$  to mean  $1 \leq x_j \leq \beta_j$ , for  $j = 1, \dots, r$ . Even further, for a positive number  $z$  we simplify and write  $\mathbf{x} \leq z$  to mean that  $1 \leq x_j \leq z$ , for  $j = 1, \dots, r$ .

We shall encounter a few times in what follows sums of the form

$$\sum_{\mathbf{x} \leq n} f(\gcd(\mathbf{x})),$$

where  $r \geq 1$  and  $f$  is some arithmetic function. They can be readily seen to be

$$(2.1) \quad \sum_{\mathbf{x} \leq n} f(\gcd(\mathbf{x})) = \sum_{j=1}^n (\mu * f)(j) \left\lfloor \frac{n}{j} \right\rfloor^r.$$

Here,  $\mu$  denotes the Möbius function and the symbol  $*$  stands for the Dirichlet convolution. The expression above is valid also for  $r = 1$ , with the conventional understanding that  $\gcd(j) = j$ , for any integer  $j \geq 1$ .

Equation (2.1) is sometimes referred as *Cesàro's formula* (see [5], [6]). Its simple proof follows:

*Proof of (2.1).* Using the properties of the Möbius function, we have that, for any arithmetical function  $F$ ,

$$(2.2) \quad \sum_{\mathbf{x} \leq n, \gcd(\mathbf{x})=1} F(\mathbf{x}) = \sum_{k=1}^n \mu(k) \sum_{\mathbf{x} \leq n, k|\mathbf{x}} F(\mathbf{x}) = \sum_{k=1}^n \mu(k) \sum_{\mathbf{y} \leq n/k} F(k\mathbf{y}).$$

In our case,

$$\begin{aligned} \sum_{\mathbf{x} \leq n} f(\gcd(\mathbf{x})) &= \sum_{d=1}^n f(d) \sum_{\mathbf{x} \leq n, \gcd(\mathbf{x})=d} 1 = \sum_{d=1}^n f(d) \sum_{\mathbf{y} \leq n/d, \gcd(\mathbf{y})=1} 1 \\ &= \sum_{kd \leq n} f(d) \mu(k) \left\lfloor \frac{n/d}{k} \right\rfloor^r = \sum_{j=1}^n \left\lfloor \frac{n}{j} \right\rfloor^r \sum_{kd=j} f(d) \mu(k). \quad \square \end{aligned}$$

We will use the following notation for the *summatory function*  $\mathcal{S}\alpha$  of an arithmetic function  $\alpha$ :

$$\mathcal{S}\alpha(x) = \sum_{j \leq x} \alpha(j) \quad \text{for each } x > 0.$$

The following arithmetic function:

$$(2.3) \quad D_r(m) = m^r - (m-1)^r, \quad m \geq 1$$

will appear several times in the paper; its summatory function is given by

$$(2.4) \quad \mathcal{S}D_r(m) = m^r.$$

Observe that  $D_r(m) \leq rm^{r-1}$ .

For each integer  $q \geq 1$ , we denote  $I_q(n) = n^q$ ,  $n \geq 1$ ; its summatory function satisfies

$$(2.5) \quad \mathcal{S}I_q(m) = \sum_{k=1}^m I_q(k) = \sum_{k=1}^m k^q = \frac{m^{q+1}}{q+1} + O_q(m^q).$$

For each integer  $q \geq 1$ , denote by  $\varphi_q$  the arithmetic function given by  $\varphi_q(n) = (\mu * I_q)(n)$  (the so-called  *$q$ -Jordan totient function*).

**Lemma 2.1.** *For integer  $q \geq 1$ , the summatory function of  $\varphi_q$  satisfies*

$$(2.6) \quad \mathcal{S}\varphi_q(n) = \begin{cases} \frac{1}{(q+1)\zeta(q+1)} n^{q+1} + O_q(n^q) & \text{if } q \geq 2, \\ \frac{1}{2\zeta(2)} n^2 + O(n \ln n) & \text{if } q = 1. \end{cases}$$

Notice that  $\varphi_1$  is just Euler's  $\varphi$  function and that the case  $q = 1$  of Lemma 2.1 is just Dirichlet's theorem.

*Proof.* Apply the well-known expression (see [1], Theorem 3.10) for the summatory function of the Dirichlet convolution of two arithmetical functions  $\alpha$  and  $\beta$ ,

$$(2.7) \quad \mathcal{S}(\alpha * \beta)(x) = \sum_{j \leq x} \alpha(j) \mathcal{S}\beta(x/j) = \sum_{j \leq x} \beta(j) \mathcal{S}\alpha(x/j).$$

and equation (2.5).  $\square$

The following is an elementary calculus lemma which shall be useful in the discussion of distributional properties of lcm.

**Lemma 2.2.** *For each  $r \geq 1$  and for all  $s > 0$ , denote by  $\Omega_r(s)$  the volume of the  $r$ -dimensional set  $\mathcal{A}_r(s) := \{(x_1, \dots, x_r) : 0 \leq x_1, \dots, x_r \leq 1, x_1 \cdots x_r \leq s\}$ . Then  $\Omega_r(s) = 1$  for  $s \geq 1$  and*

$$\Omega_r(s) = s \sum_{j=0}^{r-1} \frac{\ln(1/s)^j}{j!} \quad \text{for } s < 1.$$

*Proof.* Observe that  $\Omega_1(s) = s$  and that  $\Omega_r(s) = s + \int_s^1 \Omega_{r-1}(s/x) dx$  for  $r \geq 2$ .  $\square$

A simple change of variables gives that if  $0 \leq \beta_1, \dots, \beta_r \leq 1$ , then

$$(2.8) \quad \text{Vol}\{(x_1, \dots, x_r) : 0 \leq x_j \leq \beta_j, j = 1, \dots, r, x_1 \cdots x_r \leq s\} = B \Omega_r(s/B),$$

where  $B = \prod_{j=1}^r \beta_j$ . Observe that if  $B \leq s$ , the statement is obvious.

Later on, Section 6, while discussing waiting times, we shall need the standard Euler's extension to real argument of the harmonic numbers given by  $H_n = \sum_{j=1}^n 1/j$  for integer  $n \geq 1$ ; and the standard approximation of  $H_n$  by  $\ln(n)$ , which we record in the following:

**Lemma 2.3.** *The parametric integral defined for any real  $a > 0$  by*

$$H(a) = \int_0^\infty [1 - (1 - e^{-t})^a] dt$$

*satisfies*

$$H(a) = \ln(a) + \gamma + O\left(\frac{1}{a}\right), \quad \text{as } a \rightarrow \infty.$$

*Besides, of course, for any integer  $n \geq 1$ ,  $H(n) = H_n$ .*

### 3. PROBABILITY DISTRIBUTION OF THE GCD OF $r$ -TUPLES

In the present section, we prove Theorem A', which describes the mass function and the moments of the random variable  $\gcd(X_1^{(n)}, \dots, X_r^{(n)})$ , for  $r \geq 3$ .

Cesàro's formula (2.1), with  $f = \delta_1$ , where  $\delta_1(n) = 1$  if  $n = 1$  and is 0 elsewhere, yields

$$(3.1) \quad \mathbf{P}(\gcd(X_1^{(n)}, \dots, X_r^{(n)}) = 1) = \frac{1}{n^r} \sum_{j=1}^n \mu(j) \left\lfloor \frac{n}{j} \right\rfloor^r.$$

which readily gives that there exists a constant  $C_r > 0$  such that

$$\left| \mathbf{P}(\gcd(X_1^{(n)}, \dots, X_r^{(n)}) = 1) - \frac{1}{\zeta(r)} \right| \leq C_r \frac{1}{n} \quad \text{for any integer } n \geq 1.$$

As  $\gcd(x_1, \dots, x_r) = k$  means that  $k|x_1, \dots, k|x_r$  and  $\gcd(x_1/k, \dots, x_r/k) = 1$ , we deduce that there exists a constant  $\tilde{C}_r > 0$  such that

$$\left| \mathbf{P}(\gcd(X_1^{(n)}, \dots, X_r^{(n)}) = k) - \frac{1}{k^r \zeta(r)} \right| \leq \tilde{C}_r \frac{1}{n k^{r-1}},$$

which is statement a) of Theorem A'.

Next, we turn to moments. Statements b1) and b2) are immediate consequences of the estimate (1.6): just write

$$\mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^q) = \frac{1}{\zeta(r)} \sum_{k=1}^n \frac{k^q}{k^r} + O_r\left(\frac{1}{n} \sum_{k=1}^n \frac{k^q}{k^{r-1}}\right).$$

For  $q \leq r-2$ , the sum above tends to  $\zeta(r-q)/\zeta(r)$  as  $n \rightarrow \infty$  with an error bound  $O_r(1/n^{r-q-1})$ , which in the worst case ( $r = q+2$ ) is  $O_r(1/n)$ ; while the right hand side  $O$  term is, again in the worst case,  $O_r(\ln(n)/n)$ .

If  $q = r-1$ , we get that  $\mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^{r-1}) = \ln(n)/\zeta(r) + O_r(1)$ .



b3) The argument above would not work for  $q \geq r$ , since the purported error term turns out to be of the same order as the main term. Using Cesàro's formula (2.1), the identities (2.4) and (2.7), and Lemma 2.1, one writes

$$\begin{aligned} \mathbf{E}(\gcd(X_1^{(n)}, \dots, X_r^{(n)})^q) &= \frac{1}{n^r} \sum_{j=1}^n \varphi_q(j) \left\lfloor \frac{n}{j} \right\rfloor^r = \frac{1}{n^r} \sum_{j=1}^n D_r(j) \mathcal{S}\varphi_q(\lfloor n/j \rfloor) \\ &= \frac{1}{n^r} \frac{1}{(q+1)\zeta(q+1)} \sum_{j=1}^n D_r(j) \left\lfloor \frac{n}{j} \right\rfloor^{q+1} + O_q\left(\frac{1}{n^r} \sum_{j=1}^n D_r(j) \frac{n^q}{j^q}\right) \\ &= \frac{n^{q-r+1}}{(q+1)\zeta(q+1)} \sum_{j=1}^n \frac{D_r(j)}{j^{q+1}} + O_q\left(n^{q-r} \sum_{j=1}^n \frac{D_r(j)}{j^q}\right). \end{aligned}$$

As  $D_r(j) \leq r j^{r-1}$  and  $q \geq r \geq 3$ , this last error term is at most  $O_{q,r}(n^{q-r} \ln(n))$ . Finally, using the definition (2.3) of  $D_r(j)$ , and recalling again that  $q \geq r$ , we get that

$$\begin{aligned} \sum_{j=1}^n \frac{D_r(j)}{j^{q+1}} &= \sum_{j=1}^n \frac{j^r - (j-1)^r}{j^{q+1}} = \sum_{j=1}^n \frac{1}{j^{q+1}} \sum_{k=1}^r \binom{r}{k} (-1)^{k+1} j^{r-k} \\ &= \sum_{k=1}^r \binom{r}{k} (-1)^{k+1} \sum_{j=1}^n \frac{1}{j^{q+1-r+k}} = \sum_{k=1}^r \binom{r}{k} (-1)^{k+1} \zeta(q-r+k+1) + O_{q,r}\left(\frac{1}{n}\right), \end{aligned}$$

and this proves (1.9).

#### 4. PROBABILITY DISTRIBUTION OF THE LCM OF $r$ -TUPLES

Observe that

$$(4.1) \quad \text{lcm}(a_1, \dots, a_r) = \frac{a_1 \cdots a_r}{\prod_{i < j} \gcd(a_i, a_j)} \frac{\prod_{i < j < k} \gcd(a_i, a_j, a_k)}{\prod_{i < j < k < l} \gcd(a_i, a_j, a_k, a_l)} \cdots$$

Notice that the lcm is the product of the numbers,  $\text{lcm}(a_1, \dots, a_r) = a_1 \cdots a_r$ , if and only if they are *pairwise coprime*, that is,  $\gcd(a_i, a_j) = 1$  for each  $i \neq j$ . For  $r = 2$ , of course, there is no difference between coprimality and pairwise coprimality.

**4.1. Pairwise coprimality and equidistribution.** For a  $r$ -tuple of positive integers  $\mathbf{x}$ , we write  $\mathbf{x} \in \text{PC}$  if the  $r$  components of  $\mathbf{x}$  are pairwise coprime (that is,  $\gcd(x_i, x_j) = 1$  for  $i \neq j$ ). The following result was obtained by Toth and also by Cai–Bach (see [19] and [4]):

**Lemma 4.1.** *For each  $r \geq 2$ ,*

$$(4.2) \quad \lim_{n \rightarrow \infty} \mathbf{P}(\{X_1^{(n)}, \dots, X_r^{(n)}\} \in \text{PC}) = \prod_p \left(1 - \frac{1}{p}\right)^{r-1} \left(1 + \frac{r-1}{p}\right) := T_r.$$

When  $r = 2$ , the constant  $T_2$  is  $1/\zeta(2)$  and we recover (1.6).

Extending the argument of Cai and Bach, one could prove that the PC  $r$ -tuples are, in fact, equidistributed, in the following sense (see the details in [10]):

**Lemma 4.2.** *Fix  $r \geq 2$ . Then, for any function  $f \in C([0, 1]^r)$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} \sum_{\mathbf{x} \leq n, \mathbf{x} \in \text{PC}} f\left(\frac{x_1}{n}, \dots, \frac{x_r}{n}\right) = T_r \int_{[0, 1]^r} f(u_1, \dots, u_r) du_1 \cdots du_r.$$

Taking  $f \equiv 1$ , we recover Lemma 4.1. Let us record now two applications of Lemma 4.2 which will be useful in forthcoming arguments. Fix  $0 < t \leq 1$  and  $\beta \in \mathbb{R}^r$  with  $0 \leq \beta_1, \dots, \beta_r \leq 1$ . Let  $B = \prod_{j=1}^r \beta_j$  and define  $\mathcal{B} = [0, \beta_1] \times \dots \times [0, \beta_r]$ . Choose  $f = \mathbf{1}_{\mathcal{B} \cap \mathcal{A}_r(t)}$ , where  $\mathcal{A}_r(t)$  the region given in (1.10). Then, thanks to (2.8),

$$(4.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n^r} \# \{ \mathbf{x} \leq n\beta; \mathbf{x} \in \text{PC}; x_1 \cdots x_r \leq tn^r \} = T_r B \Omega_r(t/B),$$

where  $\Omega_r$  is the function given in (1.11).

For  $q \geq 1$ , take  $f(u_1, \dots, u_r) = u_1^q \cdots u_r^q \cdot \mathbf{1}_{\mathcal{B}}$ . Then we get

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{r+qr}} \sum_{\substack{\mathbf{x} \leq n\beta \\ \mathbf{x} \in \text{PC}}} x_1^q \cdots x_r^q = T_r \frac{1}{(q+1)^r} B^{q+1}.$$

Notice that actually the functions considered to derive these two examples are not continuous, as demanded by Lemma 4.2, but a standard approximation argument yields the results: for instance, for (4.3), consider

$$f_\varepsilon = \left( 1 - \frac{\text{dist}(\bullet, \mathcal{B} \cap \mathcal{A}_r(t))}{\varepsilon} \right)^+,$$

to let  $\varepsilon \rightarrow 0$ .

The following extension of Lemma 4.2 will be useful in Section 5. Fix an  $r$ -tuple  $\mathbf{a} = (a_1, \dots, a_r) \in \text{PC}$ . We will say that  $\mathbf{x} \in \text{PC}_{\mathbf{a}}$  if the components  $x_j$  are pairwise coprime and, additionally,  $\gcd(x_1, a_1) = \dots = \gcd(x_r, a_r) = 1$ . We refer, again, the reader to [10].

**Lemma 4.3.** *Fix  $r \geq 2$  and consider an  $r$ -tuple  $\mathbf{a} = (a_1, \dots, a_r) \in \text{PC}$ . Then, for any function  $f \in C([0, 1]^r)$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} \sum_{\mathbf{x} \leq n, \mathbf{x} \in \text{PC}_{\mathbf{a}}} f\left(\frac{x_1}{n}, \dots, \frac{x_r}{n}\right) = T_r \Upsilon_r\left(\prod_{j=1}^r a_j\right) \int_{[0, 1]^r} f(u_1, \dots, u_r) du_1 \cdots du_r,$$

where the function  $\Upsilon_r$  is given by (1.17).

Taking  $f \equiv 1$ , we obtain that the proportion of  $\text{PC}_{\mathbf{a}}$   $r$ -tuples in  $\{1, \dots, n\}^r$  tends to  $T_r \Upsilon_r(\prod_{j=1}^r a_j)$  as  $n \rightarrow \infty$ . Again, two special cases of interest:

$$(4.5) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^r} \# \{ \mathbf{x} \leq n\beta; \mathbf{x} \in \text{PC}_{\mathbf{a}}; x_1 \cdots x_r \leq tn^r \} \\ = T_r \Upsilon_r\left(\prod_{j=1}^r a_j\right) B \Omega_r(t/B), \quad \text{for } t \leq 1 \text{ fixed,} \end{aligned}$$

and for  $q \geq 1$ ,

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{r+qr}} \sum_{\mathbf{x} \leq n\beta, \mathbf{x} \in \text{PC}_{\mathbf{a}}} x_1^q \cdots x_r^q = T_r \Upsilon_r\left(\prod_{j=1}^r a_j\right) \frac{1}{(q+1)^r} B^{q+1}.$$

**4.2. Distribution function of the lcm of  $r$ -tuples.** In this subsection we shall prove Theorem 1. Fix  $r \geq 3$ .

4.2.1. *Proof of (1.13).* For  $0 \leq t \leq 1$  and real  $z \geq 1$ , we introduce the following counting functions:

$$(4.7) \quad N_r(t, z) = \#\{\mathbf{x} \leq z : x_1 \cdots x_r \leq tz^r\}$$

$$(4.8) \quad G_r(t, z) = \#\{\mathbf{x} \leq z : x_1 \cdots x_r \leq tz^r, \gcd(\mathbf{x}) = 1\}$$

$$(4.9) \quad L_r(t, z) = \#\{\mathbf{x} \leq z : \text{lcm}(\mathbf{x}) \leq tz^r\}$$

Our objective is to estimate

$$\mathbf{P}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) \leq tn^r) = \frac{L_r(t, n)}{n^r} \quad \text{as } n \rightarrow \infty,$$

and we shall use for that purpose convenient auxiliary estimates for  $N$  and  $G$ .

The following elementary lemma estimates the number of lattice points in the positive  $r$ -cube  $[1, z]^r$  such that  $x_1 \cdots x_r \leq tz^r$ , in terms of the volume of the region:

**Lemma 4.4.** *For  $0 < \delta \leq t < 1$ ,*

$$\frac{1}{z^r} N_r(t, z) = \Omega_r(t) + O_\delta\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty.$$

Observe that the bound depends on  $\delta$  and not on  $t$ . The next lemma estimates the function  $G_r(t, z)$ :

**Lemma 4.5.** a) *For  $0 < \delta \leq t < 1$ , we have that, as  $z \rightarrow \infty$ ,*

$$(4.10) \quad G_r(t, z) = \frac{\Omega_r(t)}{\zeta(r)} z^r + O_\delta(z^{r-1}).$$

b) *For  $t \geq 1$ ,*

$$(4.11) \quad G_r(t, z) = \frac{1}{\zeta(r)} z^r + O(z^{r-1}).$$

*Proof.* The estimate (4.11) is the case  $k = 1$  of (1.6), as for  $t \geq 1$ ,  $G_r(t, z)$  counts coprime  $r$ -tuples. For the case  $t < 1$ , partitioning accordingly as the value of the greatest common divisor, we can write

$$\begin{aligned} N_r(t, z) &= \sum_{d \leq z} \#\{\mathbf{x} \leq z : x_1 \cdots x_r \leq tz^r, \gcd(\mathbf{x}) = d\} \\ &= \sum_{d \leq z} \#\{\mathbf{y} \leq z/d : y_1 \cdots y_r \leq t(z/d)^r, \gcd(\mathbf{y}) = 1\} = \sum_{d \leq z} G_r(t, z/d). \end{aligned}$$

By Möbius inversion (see, for instance, Theorem 2.22 in [1]), we deduce that

$$G_r(t, z) = \sum_{d \leq z} \mu(d) N_r(t, z/d).$$

Now, for  $t \geq \delta$ , using Lemma 4.4 and  $r \geq 3$ , we get that,

$$\begin{aligned} G_r(t, z) &= \sum_{d \leq z} \mu(d) \left( \frac{z^r}{d^r} \Omega_r(t) + O_\delta\left(\frac{z^{r-1}}{d^{r-1}}\right) \right) = \Omega_r(t) z^r \sum_{d \leq z} \frac{\mu(d)}{d^r} + O_\delta(z^{r-1}) \\ &= \Omega_r(t) \frac{z^r}{\zeta(r)} + O_\delta(z^{r-1}), \end{aligned}$$

as claimed.  $\square$

Our objective is to estimate  $L_r(t, z)$ . Let us write, again by partitioning,

$$\begin{aligned} L_r(t, z) &= \sum_{d \leq z} \#\{\mathbf{x} \leq z : \text{lcm}(\mathbf{x}) \leq tz^r, \text{gcd}(\mathbf{x}) = d\} \\ &= \sum_{d \leq z} \#\{\mathbf{y} \leq z/d : \text{lcm}(\mathbf{y}) \leq td^{r-1} \left(\frac{z}{d}\right)^r, \text{gcd}(\mathbf{y}) = 1\} \\ &\geq \sum_{d \leq z} \#\{\mathbf{y} \leq z/d : y_1 \cdots y_r \leq td^{r-1} \left(\frac{z}{d}\right)^r, \text{gcd}(\mathbf{y}) = 1\} = \sum_{d \leq z} G_r(td^{r-1}, z/d), \end{aligned}$$

where we have used that if  $\text{gcd}(\mathbf{x}) = d$  and if we write  $x_i = dy_i$ , for  $i = 1, \dots, r$ , then  $\text{gcd}(\mathbf{y}) = 1$  and  $\text{lcm}(\mathbf{y}) = \text{lcm}(\mathbf{x})/d$ . We have also used that  $y_1 \cdots y_r \geq \text{lcm}(\mathbf{y})$ . The above inequality would turn into an equality if  $r = 2$ .

Therefore,

$$\mathbf{P}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}) \leq tn^r) = \frac{L_r(t, n)}{n^r} \geq \frac{1}{n^r} \sum_{d \leq n} G_r(td^{r-1}, n/d).$$

To get the lower bound (1.13), we estimate the latter sum. For that purpose, we split it into two sums, depending on whether the first argument of  $G_r$  is less than or equal to 1, or not:

$$\sum_{d \leq n} G_r(td^{r-1}, \frac{n}{d}) = \sum_{d \leq n, td^{r-1} \leq 1} G_r(td^{r-1}, \frac{n}{d}) + \sum_{d \leq n, td^{r-1} > 1} G_r(td^{r-1}, \frac{n}{d}) := \text{I} + \text{II}.$$

For  $td^{r-1} \leq 1$ , thanks to (4.10), we can write

$$G_r(td^{r-1}, \frac{n}{d}) = \frac{1}{\zeta(r)} \Omega_r(td^{r-1}) \left(\frac{n}{d}\right)^r + O_t\left(\left(\frac{n}{d}\right)^{r-1}\right)$$

where we have used, in the  $O$  term, that  $td^{r-1} \geq t$  since  $d \geq 1$ . So the term I can be written as

$$\text{I} = \frac{n^r}{\zeta(r)} \sum_{d \leq n, td^{r-1} \leq 1} \frac{\Omega_r(td^{r-1})}{d^r} + O_t(n^{r-1}),$$

where we have used that  $r \geq 3$ .

On the other hand, using now (4.11) and  $r \geq 3$ ,

$$\text{II} = \sum_{d \leq n, td^{r-1} > 1} \frac{1}{\zeta(r)} \left(\frac{n}{d}\right)^r + O\left(\sum_{d \leq n} \left(\frac{n}{d}\right)^{r-1}\right) = \frac{n^r}{\zeta(r)} \sum_{d \leq n, td^{r-1} > 1} \frac{1}{d^r} + O(n^{r-1}).$$

Adding up the expressions for I and II,

$$\begin{aligned} \sum_{d \leq n} G_r(td^{r-1}, \frac{n}{d}) &= \frac{n^r}{\zeta(r)} \sum_{d \leq n} \frac{1}{d^r} + \frac{n^r}{\zeta(r)} \sum_{d \leq n, td^{r-1} \leq 1} \frac{(\Omega_r(td^{r-1}) - 1)}{d^r} + O_t(n^{r-1}) \\ &= n^r \left(1 - \frac{1}{\zeta(r)} \sum_{d \leq n, td^{r-1} \leq 1} \frac{(1 - \Omega_r(td^{r-1}))}{d^r}\right) + O_t(n^{r-1}), \end{aligned}$$

from which inequality (1.13) is proved.

4.2.2. *Proof of (1.14).* We now obtain a lower bound of the complementary probability by restricting to those  $r$ -tuples whose pairwise gcd are all equal and then partitioning according to the value of that common gcd. Fix  $0 \leq t \leq 1$ . Observe that

$$\#\{\mathbf{x} \leq n : \text{lcm}(\mathbf{x}) > tn^r\} \geq \sum_{k=1}^n \#\{\mathbf{x} \leq n : \text{lcm}(\mathbf{x}) > tn^r, \gcd(x_i, x_j) = k \text{ for } i \neq j\}.$$

Writing  $x_i = ky_i$ , for each  $i = 1, \dots, r$ , we get that  $\mathbf{y} \in \text{PC}$  and that  $\text{lcm}(\mathbf{x}) = k \cdot y_1 \cdots y_r$ , and conversely. So we can write

$$\begin{aligned} \frac{1}{n^r} \#\{\mathbf{x} \leq n : \text{lcm}(\mathbf{x}) > tn^r\} &\geq \frac{1}{n^r} \sum_{k=1}^n \#\{\mathbf{y} \leq \frac{n}{k} : y_1 \cdots y_r > tk^{r-1}(\frac{n}{k})^r, \mathbf{y} \in \text{PC}\} \\ &= \sum_{tk^{r-1} < 1} \frac{1}{k^r} \frac{1}{(n/k)^r} \#\{\mathbf{y} \leq \frac{n}{k} : y_1 \cdots y_r > tk^{r-1}(\frac{n}{k})^r, \mathbf{y} \in \text{PC}\}. \end{aligned}$$

For fixed  $t$ , the sum has a finite number of terms, and the  $k^{\text{th}}$  one tends to

$$\frac{1}{k^r} T_r(1 - \Omega_r(tk^{r-1})), \quad \text{as } n \rightarrow \infty,$$

according to (4.3). So the whole sum tends

$$T_r \sum_{tk^{r-1} < 1} \frac{1 - \Omega_r(tk^{r-1})}{k^r}, \quad \text{as } n \rightarrow \infty,$$

and (1.14) is proved.

4.3. **Moments of the lcm of  $r$ -tuples.** In this subsection, we shall prove Theorem 2, with an argument akin to that in Theorem 10 of [13].

4.3.1. *Proof of (1.15).* The following lemma is a direct application of (2.2):

**Lemma 4.6.** *Fix  $q \geq 1$  and consider the summatory function  $\mathcal{S}I_q(n) = \sum_{j \leq n} j^q$ . Then, for  $r \geq 2$ ,*

$$(4.12) \quad \sum_{\substack{\mathbf{y} \leq n, \\ \gcd(\mathbf{y})=1}} y_1^q \cdots y_r^q = \sum_{d \leq n} \mu(d) d^{rq} [\mathcal{S}I_q(n/d)]^r.$$

Applying this lemma and the trivial estimate  $\text{lcm}(\mathbf{y}) \leq y_1 \cdots y_r$  we get

$$\begin{aligned} \sum_{\mathbf{x} \leq n} \text{lcm}(\mathbf{x})^q &= \sum_{d \leq n} \sum_{\substack{\mathbf{x} \leq n, \\ \gcd(\mathbf{x})=d}} \text{lcm}(\mathbf{x})^q = \frac{1}{n^r} \sum_{d \leq n} d^q \sum_{\substack{\mathbf{y} \leq n/d, \\ \gcd(\mathbf{y})=1}} \text{lcm}(\mathbf{y})^q \\ (4.13) \quad &\leq \sum_{kd \leq n} d^q \mu(k) k^{rq} \left[ \mathcal{S}I_q\left(\frac{n}{dk}\right) \right]^r. \end{aligned}$$

This means that

$$\mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})^q) \leq \frac{1}{n^r} \sum_{kd \leq n} d^q \mu(k) k^{rq} \left[ \mathcal{S}I_q\left(\frac{n}{dk}\right) \right]^r,$$

and we will get (1.15) by estimating this last sum. From (2.5), we deduce that

$$\left[ \mathcal{S}I_q\left(\frac{n}{dk}\right) \right]^r = \frac{1}{(q+1)^r} \left(\frac{n}{dk}\right)^{r(q+1)} + O_{q,r}\left(\frac{n}{dk}\right)^{r(q+1)-1}.$$

Plugging this into (4.13), the sum of leading terms is given by

$$n^{rq} \frac{1}{(q+1)^r} \frac{\zeta(r(q+1)-q)}{\zeta(r)} + O_{q,r}(n^{r(q-1)+1}),$$

while the sum of the error terms is seen to be  $O_{q,r}(n^{rq-1})$ . Adding this up, we obtain (1.15).

4.3.2. *Proof of (1.16).* Arguing as in the proof of (1.14),

$$\frac{1}{n^{rq+r}} \sum_{\mathbf{x} \leq n} \text{lcm}(\mathbf{x})^q \geq \frac{1}{n^{rq+r}} \sum_{k=1}^n \sum_{\substack{\mathbf{x} \leq n \\ \gcd(x_i, x_j)=k, i \neq j}} \text{lcm}(\mathbf{x})^q = \sum_{k=1}^n \frac{k^q}{n^{rq+r}} \sum_{\substack{\mathbf{y} \leq n/k \\ \mathbf{y} \in \text{PC}}} y_1^q \cdots y_r^q.$$

According to (4.4) (with  $\beta_j = 1/k$ ), for each summand we have that

$$\lim_{n \rightarrow \infty} \frac{k^q}{n^{rq+r}} \sum_{\substack{\mathbf{y} \leq n/k \\ \mathbf{y} \in \text{PC}}} y_1^q \cdots y_r^q = \frac{T_r}{(q+1)^r} \frac{1}{k^{r(q+1)-q}}$$

From the bound

$$\frac{k^q}{n^{rq+r}} \sum_{\substack{\mathbf{y} \leq n/k \\ \mathbf{y} \in \text{PC}}} y_1^q \cdots y_r^q \leq \frac{k^q}{n^{rq+r}} \left(\frac{n}{k}\right)^{rq} \left(\frac{n}{k}\right)^r = \frac{1}{k^{r(q+1)-q}},$$

dominated convergence and the fact that  $\sum_{k=1}^{\infty} 1/k^{r(q+1)-q} < +\infty$ , we deduce

$$\lim_{n \rightarrow \infty} \frac{1}{n^{rq+r}} \sum_{k=1}^n \sum_{\substack{\mathbf{x} \leq n \\ \gcd(x_i, x_j)=k, i \neq j}} \text{lcm}(\mathbf{x})^q = \frac{T_r}{(q+1)^r} \zeta(r(q+1)-q),$$

and therefore,

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{rq}} \mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})) = \liminf_{n \rightarrow \infty} \frac{1}{n^{rq+r}} \sum_{\mathbf{x} \leq n} \text{lcm}(\mathbf{x})^q \geq \frac{T_r \zeta(r(q+1)-q)}{(q+1)^r}.$$

This proves (1.16).

4.4. **The logarithm of the lcm.** The structure of the lcm exhibited in (4.1) invites to consider its logarithm, as it may be written in terms of sums of logarithms of gcd's of different lengths:

$$(4.14) \quad \begin{aligned} \ln(\text{lcm}(a_1, \dots, a_r)) &= \sum_{j=1}^r \ln(a_j) - \sum_{i < j} \ln(\gcd(a_i, a_j)) \\ &\quad + \sum_{i < j < k} \ln(\gcd(a_i, a_j, a_k)) - \cdots \end{aligned}$$

Using this, we can prove the following:

**Proposition 4.7.** *For  $r \geq 2$ ,*

$$(4.15) \quad \lim_{n \rightarrow \infty} \left[ \mathbf{E}(\ln(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}))) - \frac{r}{n} \sum_{j=1}^n \ln(j) \right] = \sum_{k=2}^r \binom{r}{k} (-1)^k \frac{\zeta'(k)}{\zeta(k)}.$$

*Proof.* Observe that, by Cesàro's formula (2.1),

$$\begin{aligned} \mathbf{E}(\ln(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)}))) &= \binom{r}{1} \mathbf{E}(\ln(X_1^{(n)})) - \binom{r}{2} \mathbf{E}(\ln(\gcd(X_1^{(n)}, X_2^{(n)}))) + \dots \\ &= \frac{r}{n} \sum_{j=1}^n \ln(j) - \sum_{k=2}^r \binom{r}{k} (-1)^k \sum_{j=1}^n (\mu * \ln)(j) \left( \left\lfloor \frac{n}{k} \right\rfloor \frac{1}{n} \right)^k \\ &= \frac{r}{n} \sum_{j=1}^n \ln(j) - \sum_{k=2}^r \binom{r}{k} (-1)^k \sum_{j=1}^n \Lambda(j) \left( \left\lfloor \frac{n}{k} \right\rfloor \frac{1}{n} \right)^k. \end{aligned}$$

We have used the fact that  $(\mu * \ln)(j) = \Lambda(j)$ , where  $\Lambda$  denotes the von Mangoldt's function (see Theorem 295 in [14]). For  $k \geq 2$  fixed,

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n \Lambda(j) \left( \left\lfloor \frac{n}{j} \right\rfloor \frac{1}{n} \right)^k = \sum_{j=1}^{\infty} \frac{\Lambda(j)}{j^k} = -\frac{\zeta'(k)}{\zeta(k)},$$

where we have used the trivial estimate  $\Lambda(n) \leq \ln(n)$  and the standard expression of the Dirichlet series of the function  $\Lambda$  (see Theorem 294 in [14]).  $\square$

By the way, from (4.15) and Jensen's inequality one gets that

$$\liminf_{n \rightarrow \infty} \frac{\mathbf{E}(\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})^q)}{n^{rq}} \geq e^{q(H(r)-r)},$$

where  $H(r)$  is the function defined in the right hand side of (4.15), a lower bound weaker than (1.16), but of the proper order.

**4.5. Alternative normalization of lcm.** As we have mentioned before, one may alternatively normalize lcm by dividing it by the product of the numbers. With an argument similar, but simpler, to the one used to prove Theorem 1, one may derive:

**Proposition 4.8.** *For  $r \geq 2$ , and for every  $0 < t \leq 1$ ,*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbf{P}\left(\frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{X_1^{(n)} \dots X_r^{(n)}} \leq t\right) &\geq 1 - \frac{1}{\zeta(r)} \sum_{j^{r-1} < 1/t} \frac{1}{j^r}, \\ \limsup_{n \rightarrow \infty} \mathbf{P}\left(\frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{X_1^{(n)} \dots X_r^{(n)}} \leq t\right) &\leq 1 - T_r \sum_{j^{r-1} < 1/t} \frac{1}{j^r}. \end{aligned}$$

Actually, for  $r = 2$ , we have equality for every  $0 < t \leq 1$ :

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{\text{lcm}(X_1^{(n)}, X_2^{(n)})}{X_1^{(n)} X_2^{(n)}} \leq t\right) = 1 - \frac{1}{\zeta(2)} \sum_{j < 1/t} \frac{1}{j^2}.$$

Of course, the statement above for  $r = 2$  gives the asymptotic behavior of the distribution function of  $1/\gcd(X_1^{(n)}, X_2^{(n)})$ . The limiting distribution is discrete: it assigns mass  $\frac{1}{\zeta(2)k^2}$  to the point  $1/k$ , for every integer  $k \geq 1$ ; in contrast to the limit distribution in Theorem B, part (a), which has no point masses.

For moments, we have:

**Proposition 4.9.** *For  $r \geq 2$ , and integer  $q \geq 1$ ,*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{E} \left( \left( \frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{X_1^{(n)} \dots X_r^{(n)}} \right)^q \right) &\leq \frac{1}{\zeta(r)} \zeta(r(q+1) - q), \\ \liminf_{n \rightarrow \infty} \mathbf{E} \left( \left( \frac{\text{lcm}(X_1^{(n)}, \dots, X_r^{(n)})}{X_1^{(n)} \dots X_r^{(n)}} \right)^q \right) &\geq T_r \zeta(r(q+1) - q). \end{aligned}$$

Actually, for  $r = 2$ , we have equality for every integer  $q \geq 1$ :

$$\lim_{n \rightarrow \infty} \mathbf{E} \left( \left( \frac{\text{lcm}(X_1^{(n)}, X_2^{(n)})}{X_1^{(n)} \cdot X_2^{(n)}} \right)^q \right) = \frac{1}{\zeta(2)} \zeta(q+2).$$

## 5. THE CASE $r = 3$

The analysis of this case  $r = 3$  rests on a particular partition of the triples of integers which is based in the following factorization lemma:

**Lemma 5.1.** *To any triple of integers  $(x, y, z)$  we may assign uniquely an integer  $D$  and triples of integers  $(a, b, c) \in \text{PC}$  and  $(u, v, w) \in \text{PC}_{(c, b, a)}$  so that*

$$x = D(ab)u, \quad y = D(ac)v, \quad z = D(bc)w.$$

In fact,  $D = \gcd(x, y, z)$ ,

$$(5.1) \quad \begin{cases} a = \gcd(x, y)/D, \\ b = \gcd(x, z)/D, \\ c = \gcd(y, z)/D, \end{cases} \quad \text{and} \quad \begin{cases} u = x/(Dab), \\ v = y/(Dac), \\ w = z/(Dbc). \end{cases}$$

Moreover,

$$\text{lcm}(x, y, z) = D(abc)(uvw).$$

The proof is direct; we just insist that  $(u, c), (v, b), (w, a)$  are required to be coprime couples. For pairs of integers  $(x, y)$  the analogous representation is  $x = Du, y = Dv$ , with  $D = \gcd(x, y)$  and  $u, v$  coprime.

**5.1. Proof of part a) of Theorem 3.** Fix an integer  $n \geq 1$  and  $0 < t \leq 1$ . We follow Lemma 5.1 and partition the required counting:

$$\begin{aligned} &\#\{1 \leq x, y, z \leq n; \text{lcm}(x, y, z) > tn^3\} \\ &= \sum_{D=1}^{\infty} \sum_{(a, b, c) \in \text{PC}} \# \left\{ \begin{array}{l} u \leq n/(Dab) \\ v \leq n/(Dac) : (u, v, w) \in \text{PC}_{(c, b, a)}, \quad uvw > t \frac{n^3}{Dabc} \\ w \leq n/(Dbc) \end{array} \right\} \end{aligned}$$

Now, according to (4.5), the argument of this double sum, for fixed  $D$  and fixed  $a, b, c$ , is asymptotically

$$\sim n^3 T_3 \Upsilon_3(abc) \frac{1}{D^3(abc)^2} \left( 1 - \Omega_3(tD^2abc) \right)$$

and is bounded by

$$\leq n^3 \frac{1}{D^3(abc)^2}$$



Since

$$\sum_{D=1}^{\infty} \sum_{(a,b,c) \in \text{PC}} \frac{1}{D^3(abc)^2} = \zeta(3) \sum_{(a,b,c) \in \text{PC}} \frac{1}{(abc)^2} = \zeta(3) \sum_{m=1}^{\infty} \frac{3^{\omega(m)}}{m^2} < +\infty,$$

dominated convergence gives the result. Here we have used that for any given integer  $m$  there are exactly  $3^{\omega(m)}$  triples  $(a, b, c) \in \text{PC}$  such that  $m = abc$ .

**Remark 5.2.** For  $t = 0$ , equation (1.19) reads  $T_3 \zeta(3) J(2) = 1$ . This begs to be proved directly. Observe that the function  $\tilde{\Upsilon}_3(m) = \Upsilon_3(m) 3^{\omega(m)}$  is multiplicative, and that, for any prime  $p$  and any positive integer  $a$ ,  $\hat{\Upsilon}_3(p^a) = \hat{\Upsilon}_3(p) = 3^{\frac{1+1/p}{1+2/p}}$ . We can write the Dirichlet series defined in (1.18) as a product over primes:

$$J(s) = \sum_{m=1}^{\infty} \frac{\tilde{\Upsilon}_3(m)}{m^s} = \prod_p \left( 1 + \frac{\tilde{\Upsilon}_3(p)}{p^s} + \frac{\tilde{\Upsilon}_3(p^2)}{p^{2s}} + \cdots \right) = \prod_p \left( 1 + \frac{3(p+1)}{(p+2)(p^s-1)} \right),$$

so

$$J(2) = \prod_p \left( 1 + \frac{3}{(p+2)(p-1)} \right) = \prod_p \left( \frac{p^2+p+1}{(p+2)(p-1)} \right).$$

The reader may check, from the definition of  $T_3$  in (4.2) and the Euler product expression for  $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ , that  $T_3 \zeta(3) = \prod_p \frac{(p-1)(p+2)}{p^2+p+1}$ .

**5.2. Proof of part b) of Theorem 3.** Fix  $q \geq 1$ . Lemma 5.1 allow us to write

$$\sum_{x,y,z \leq n} \text{lcm}(x,y,z)^q = \sum_{D=1}^{\infty} \sum_{(a,b,c) \in \text{PC}} \left( D^q(abc)^q \sum'_{D; a,b,c} (uvw)^q \right),$$

where for  $D$  and  $(a, b, c)$  fixed, the corresponding sum  $\sum'_{D; a,b,c}$  extends over

$$\{(u, v, w) \in \text{PC}, u \leq n/[D(ab)], v \leq n/[D(ac)], w \leq n/[D(bc)]\}.$$

By (4.6), each  $\sum'_{D; a,b,c}$  is seen to be, asymptotically

$$\sum'_{D; a,b,c} \sim n^{3q+3} T_3 \Upsilon_3(abc) \frac{1}{(q+1)^3} \left( \frac{1}{D^3(abc)^2} \right)^{q+1}, \quad \text{as } n \rightarrow \infty,$$

and is bounded by

$$\sum'_{D; a,b,c} \leq n^{3q+3} \left( \frac{1}{D^3(abc)^2} \right)^{q+1}.$$

Since

$$\sum_{D=1}^{\infty} \sum_{(a,b,c) \in \text{PC}} D^q(abc)^q \left( \frac{1}{D^3(abc)^2} \right)^{q+1} = \zeta(2q+3) \sum_{m=1}^{\infty} \frac{3^{\omega(m)}}{m^{q+2}} < \infty,$$

dominated convergence gives that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{3q+3}} \sum_{x,y,z \leq n} \text{lcm}(x,y,z)^q = T_3 \frac{1}{(q+1)^3} \zeta(2q+3) \sum_{m=1}^{\infty} \frac{\Upsilon_3(m) 3^{\omega(m)}}{m^{q+2}},$$

as claimed in (1.20).

**5.3. Case  $r = 3$ , with alternative normalization.** We have for every  $0 < t \leq 1$ ,

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\left(\frac{\text{lcm}(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})}{X_1^{(n)} X_2^{(n)} X_3^{(n)}}\right) \leq t\right) = T_3 \sum_{D^2 m \geq 1/t} \frac{1}{D^3} \frac{\Upsilon_3(m) 3^{\omega(m)}}{m^2}.$$

Also, for integer  $q \geq 1$ :

$$\lim_{n \rightarrow \infty} \mathbf{E}\left(\left(\frac{\text{lcm}(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})}{X_1^{(n)} X_2^{(n)} X_3^{(n)}}\right)^q\right) = T_3 \zeta(2q+3) J(2+q).$$

## 6. WAITING TIMES

Fix  $n \geq 2$  and consider the following experiment: draw integers  $x_1, x_2, \dots$  uniformly and independently from  $\{1, \dots, n\}$ , and calculate the sequences of successive gcds and lcms:

$$\begin{aligned} z_1 &= x_1, \quad z_2 = \gcd(x_1, x_2), \quad z_3 = \gcd(x_1, x_2, x_3), \dots \\ w_1 &= x_1, \quad w_2 = \text{lcm}(x_1, x_2), \quad w_3 = \text{lcm}(x_1, x_2, x_3), \dots \end{aligned}$$

The sequence  $(z_j)$  is decreasing, and each  $z_j \geq 1$ , while the sequence  $(w_j)$  is increasing and each  $w_j \leq \text{lcm}(1, \dots, n)$ .

For each of these random sequences we are interested in the random variable that registers the first time when they reach their respective limiting values. Again, the case of the lcm is quite more involved than the case of the gcd.

**6.1. Waiting time for the gcd.** For fixed  $n$ , consider the (decreasing) sequence  $(\mathcal{Z}_m^{(n)})$  of random variables given by

$$\mathcal{Z}_1^{(n)} = X_1^{(n)}, \quad \mathcal{Z}_m^{(n)} = \gcd(\mathcal{Z}_{m-1}^{(n)}, X_m^{(n)}) = \gcd(X_1^{(n)}, \dots, X_m^{(n)}) \quad \text{for } m \geq 2;$$

Denote by  $\mathcal{T}_n$  the first time when the sequence  $(\mathcal{Z}_m^{(n)})$  reaches the value 1. The variable  $\mathcal{T}_n$  takes values  $1, 2, \dots$

**Lemma 6.1.** a) For fixed  $n$ ,  $\mathbf{P}(\mathcal{Z}_m^{(n)} = 1) \rightarrow 1$  as  $m \rightarrow \infty$ .

b) The mass function of  $\mathcal{T}_n$  is given by

$$(6.1) \quad \mathbf{P}(\mathcal{T}_n > m) = - \sum_{k=2}^n \mu(k) \left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^m \quad \text{for } m \geq 1.$$

and, for each  $m \geq 1$ ,

$$(6.2) \quad \lim_{n \rightarrow \infty} \mathbf{P}(\mathcal{T}_n > m) = 1 - \frac{1}{\zeta(m)}.$$

Notice that the case  $m = 1$  of (6.1) reduces to  $\mathbf{P}(\mathcal{T}_n > 1) = 1 - 1/n$ . The case  $m = 1$  of (6.2) is then obvious.

*Proof.* a) Observe that  $\mathbf{P}(\mathcal{Z}_1^{(n)} = 1) = 1/n$ . Recall from (3.1) that, for  $m \geq 2$ ,

$$\mathbf{P}(\mathcal{Z}_m^{(n)} = 1) = \mathbf{P}(\gcd(X_1^{(n)}, \dots, X_m^{(n)}) = 1) = \frac{1}{n^m} \sum_{k=1}^n \mu(k) \left\lfloor \frac{n}{k} \right\rfloor^m,$$

so

$$|1 - \mathbf{P}(\mathcal{Z}_m^{(n)} = 1)| \leq \sum_{k=2}^{\infty} \frac{1}{k^m} \leq \frac{3}{2^m}$$

and therefore,  $\mathbf{P}(\mathcal{Z}_m^{(n)} = 1) \rightarrow 1$  as  $m \rightarrow \infty$  (for fixed  $n$ ).

b) Observe that, for each  $m \geq 1$ , the events  $\{\mathcal{T}_n \leq m\}$  and  $\{\mathcal{Z}_m^{(n)} = 1\}$  coincide, and therefore,

$$\mathbf{P}\{\mathcal{T}_n > m\} = 1 - \mathbf{P}\{\mathcal{T}_n \leq m\} = 1 - \frac{1}{n^m} \sum_{k=1}^n \mu(k) \left\lfloor \frac{n}{k} \right\rfloor^m = - \sum_{k=2}^n \mu(k) \left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^m.$$

From here one deduces immediately that, for  $m \geq 2$ ,

$$\left| \mathbf{P}(\mathcal{T}_n > m) - \left( 1 - \frac{1}{\zeta(m)} \right) \right| = O\left( \frac{\ln(n)}{n} \right).$$

This gives (6.2) for each  $m \geq 2$ .  $\square$

The above result tells us that  $\mathcal{T}_n$  converges in distribution to  $\mathcal{T}$ , where  $\mathcal{T}$  is the random variable given by  $\mathbf{P}(\mathcal{T} \leq m) = 1/\zeta(m)$ . Formulas for the expected values of these variables,  $\mathcal{T}_n$  and  $\mathcal{T}$ , are given in the following result:

**Theorem 6.2.**

$$(6.3) \quad \lim_{n \rightarrow \infty} \mathbf{E}(\mathcal{T}_n) = 1 - \sum_{k=2}^{\infty} \frac{\mu(k)}{k-1} = \mathbf{E}(\mathcal{T}) = 2 + \sum_{m=2}^{\infty} \left( 1 - \frac{1}{\zeta(m)} \right).$$

The numerical value of  $\mathbf{E}(\mathcal{T})$  is around 2,7052; by the way, the sequence  $\mathbf{E}(\mathcal{T}_n)$  is not increasing.

*Proof.* Changing the summation order,

$$\begin{aligned} \mathbf{E}(\mathcal{T}_n) &= \sum_{m=0}^{\infty} \mathbf{P}(\mathcal{T}_n > m) = 2 - \frac{1}{n} - \sum_{m=2}^{\infty} \sum_{k=2}^n \mu(k) \left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^m \\ &= 2 - \frac{1}{n} - \sum_{k=2}^n \mu(k) \sum_{m=2}^{\infty} \left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^m = 2 - \frac{1}{n} - \sum_{k=2}^n \mu(k) \frac{\left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^2}{1 - \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor}. \end{aligned}$$

Now observe that

$$\frac{\left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^2}{1 - \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor} \leq \frac{1}{k(k-1)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\left( \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor \right)^2}{1 - \frac{1}{n} \left\lfloor \frac{n}{k} \right\rfloor} = \frac{1}{k(k-1)}.$$

As  $\sum_{k \geq 2} 1/(k(k-1)) = 1$ , by dominated convergence,

$$\lim_{n \rightarrow \infty} \mathbf{E}(\mathcal{T}_n) = 2 - \sum_{k=2}^{\infty} \frac{\mu(k)}{k(k-1)} = 2 - \sum_{k=2}^{\infty} \mu(k) \left( \frac{1}{k-1} - \frac{1}{k} \right) = 1 - \sum_{k=2}^{\infty} \frac{\mu(k)}{k-1},$$

where we have used Landau's classical result (see [17]) that  $\sum_{k=1}^{\infty} \mu(k)/k = 0$ . It can be seen (again by dominated convergence) that this sum coincides with

$$\mathbf{E}(\mathcal{T}) = 2 + \sum_{m=2}^{\infty} \left( 1 - \frac{1}{\zeta(m)} \right). \quad \square$$

**6.2. Waiting for the lcm.** Consider now the (increasing) sequence  $(\mathcal{W}_m^{(n)})$  of random variables given by

$$\mathcal{W}_1^{(n)} = X_1^{(n)}, \quad \mathcal{W}_m^{(n)} = \text{lcm}(\mathcal{W}_{m-1}^{(n)}, X_m^{(n)}) = \text{lcm}(X_1^{(n)}, \dots, X_m^{(n)}) \quad \text{for } m \geq 2;$$

Let us denote again by  $\mathcal{T}_n$  the first time when the sequence  $(\mathcal{W}_m^{(n)})$  reaches its limiting value,  $\text{lcm}(1, \dots, n)$ .

In the analysis of  $\mathcal{T}_n$  we will use the following notation: for each prime  $p \leq n$ , denote by  $\gamma_p(n)$  the integer such that

$$(6.4) \quad p^{\gamma_p(n)} \leq n \quad \text{and} \quad p^{\gamma_p(n)+1} > n.$$

In formula,  $\gamma_p(n) = \lfloor \ln n / \ln p \rfloor$ . Notice that

$$\text{lcm}(1, \dots, n) = \prod_{p \leq n} p^{\gamma_p(n)}.$$

Now define  $\beta_p(n)$  as the positive integer satisfying

$$(6.5) \quad p^{\gamma_p(n)} \beta_p(n) \leq n \quad \text{and} \quad p^{\gamma_p(n)} (\beta_p(n) + 1) > n,$$

so that  $\beta_p(n) = \lfloor n / p^{\gamma_p(n)} \rfloor$ . Observe that  $1 \leq \beta_p(n) < p$ .

For each prime  $p \leq n$ , we consider the set  $\mathcal{C}_p(n)$  of integers given by

$$\mathcal{C}_p(n) = \{p^{\gamma_p(n)}, 2p^{\gamma_p(n)}, \dots, \beta_p(n) p^{\gamma_p(n)}\}.$$

It is immediate to check that, for fixed  $n$ , the sets  $\mathcal{C}_p(n)$  and  $\mathcal{C}_q(n)$  are disjoint if  $p$  and  $q$  are different primes. Finally, call

$$\mathcal{C}^*(n) = \{1, \dots, n\} - \bigcup_{p \leq n} \mathcal{C}_p(n).$$

Observe that the event of interest,  $\{\text{lcm}(X_1^{(n)}, \dots, X_m^{(n)}) = \text{lcm}(1, \dots, n)\}$ , can be written as

$$\{\text{for each } p \leq n, \text{ at least one among } X_1^{(n)}, \dots, X_m^{(n)} \text{ belongs to } \mathcal{C}_p(n)\}.$$

This observation allows us to rewrite the waiting time question as a “weighted” coupon collector problem: we draw coupons (independently) from  $\{1, \dots, n\}$ , and to complete the collection of interest means to get, at least, one coupon from each of the classes  $\mathcal{C}_p(n)$ ,  $p \leq n$ . There are  $\pi(n)$  different classes, where  $\pi(n)$  denotes the number of primes  $\leq n$ , each one of them having “weight”  $\beta_p(n)/n$ . The coupons from the set  $\mathcal{C}^*(n)$  are useless for our objective. See, for instance, [12], [3], or the survey [2] for information about a variety of coupon collector problems.

In this language, the variable  $\mathcal{T}_n$  registers the time when the coupon collection is completed. Observe that  $\mathbf{P}(\mathcal{T}_n > l) = 1$  if  $l < \pi(n)$ . In general,

$$\{\mathcal{T}_n > l\} = \bigcup_{p \leq n} A_p(l),$$

where  $A_p(l)$  is the event in which, among the first  $l$  coupons drawn, none of them belongs to  $\mathcal{C}_p(n)$ . Applying the inclusion/exclusion principle, we can write

$$\begin{aligned} \mathbf{P}(\mathcal{T}_n > l) &= \sum_{p \leq n} \mathbf{P}(A_p(l)) - \sum_{p < q \leq n} \mathbf{P}(A_p(l) \cap A_q(l)) + \dots \\ &= \sum_{p \leq n} \left(1 - \frac{\beta_p(n)}{n}\right)^l - \sum_{p < q \leq n} \left(1 - \frac{\beta_p(n) + \beta_q(n)}{n}\right)^l + \dots \end{aligned}$$

From this expression for the distribution function of  $\mathcal{T}_n$ , and following [12], we get a compact formula for the expected waiting time:  $\mathcal{T}_n$ :

**Lemma 6.3.** *For fixed  $n$ ,*

$$(6.6) \quad \mathbf{E}(\mathcal{T}_n) = n \int_0^\infty \left[1 - \prod_{p \leq n} (1 - e^{-t \beta_p(n)})\right] dt.$$

*Proof.* Write

$$\begin{aligned} \mathbf{E}(\mathcal{T}_n) &= \sum_{l=0}^{\infty} \mathbf{P}(\mathcal{T}_n > l) = \sum_{l=0}^{\infty} \left( \sum_{p \leq n} \left(1 - \frac{\beta_p(n)}{n}\right)^l - \sum_{p \neq q} \left(1 - \frac{\beta_p(n) + \beta_q(n)}{n}\right)^l + \dots \right) \\ &= \sum_{p \leq n} \sum_{l=0}^{\infty} \left(1 - \frac{\beta_p(n)}{n}\right)^l - \sum_{p < q \leq n} \sum_{l=0}^{\infty} \left(1 - \frac{\beta_p(n) + \beta_q(n)}{n}\right)^l + \dots \\ &= n \left( \sum_{p \leq n} \frac{1}{\beta_p(n)} - \sum_{p < q \leq n} \frac{1}{\beta_p(n) + \beta_q(n)} + \dots \right). \end{aligned}$$

The identity (6.6) now follows.  $\square$

If we were to care just for the specific coupons  $\{p^{\gamma_p(n)}\}_{p \leq n}$  (the first coupon in each class  $\mathcal{C}_p(n)$ ), then the time  $\tilde{\mathcal{T}}_n$  to collect all of them will be, of course, longer than the time  $\mathcal{T}_n$ ; and, in particular, on average:  $\mathbf{E}(\tilde{\mathcal{T}}_n) \geq \mathbf{E}(\mathcal{T}_n)$  (see later (6.16) for a precise comparison). In this case there are exactly  $\pi(n)$  coupons of interest (with weight 1) out of a total on  $n$  coupons and, therefore, see Lemma 2.3,

$$\mathbf{E}(\tilde{\mathcal{T}}_n) = n \int_0^{\infty} \left[ 1 - \prod_{p \leq n} (1 - e^{-t}) \right] dt = n \int_0^{\infty} 1 - (1 - e^{-t})^{\pi(n)} dt = n H_{\pi(n)}.$$

The asymptotic size of  $\mathbf{E}(\tilde{\mathcal{T}}_n)$  may be obtained by appealing to the elementary Lemma 2.3 and to the standard error bound on the Prime Number Theorem:

$$(6.7) \quad \left| \pi(n) - \frac{n}{\ln(n)} \right| \leq C \frac{n}{\ln(n)^2},$$

valid for each  $n \geq 1$  ( $C$  is an absolute constant). We may write:

$$\begin{aligned} (6.8) \quad \mathbf{E}(\tilde{\mathcal{T}}_n) &= n H_{\pi(n)} = n (\ln(\pi(n)) + \gamma + O(\ln(n)/n)) \\ &= n \ln(n) - n \ln \ln(n) + n\gamma + O(n/\ln(n)). \end{aligned}$$

We would like to obtain an asymptotic expression like (6.8), but for  $\mathcal{T}_n$ . For that purpose, we introduce the following frequency counting functions of the  $\beta_p(n)$ :

$$(6.9) \quad \omega_j(n) = \#\{p \leq n : \beta_p(n) = j\}, \quad j \geq 1.$$

A few properties of these  $\omega_j$  are in order.

**Lemma 6.4.** a)  $\omega_j(n) = 0$ , if  $j \geq \sqrt{n}$ .

$$\text{b) } \sum_{j=1}^{\infty} \omega_j(n) = \pi(n).$$

$$\text{c) For } j \geq 1 \text{ such that } j+1 \leq \sqrt{n},$$

$$(6.10) \quad \pi\left(\frac{n}{j}\right) - \pi\left(\frac{n}{j+1}\right) \leq \omega_j(n) \leq \pi\left(\frac{n}{j}\right) - \pi\left(\frac{n}{j+1}\right) + \pi(\sqrt{n}).$$

$$\text{d) } \sum_{j=1}^{\infty} j \omega_j(n) = \sum_{p \leq n} \beta_p(n) \sim n \ln(2), \text{ as } n \rightarrow \infty. \text{ Actually,}$$

$$(6.11) \quad \frac{1}{n} \sum_{p \leq n} \beta_p(n) = \ln(2) + O\left(\frac{1}{\ln(n)}\right), \quad \text{as } n \rightarrow \infty.$$

$$\text{e) For each } j \geq 1,$$

$$(6.12) \quad \lim_{n \rightarrow \infty} \frac{\omega_j(n)}{\pi(n)} = \frac{1}{j(j+1)}.$$

Observe that part d) means that asymptotically the proportion of useless coupons out of the total of  $n$  coupons (the set  $\mathcal{C}_n^*$ ) is  $1 - \ln(2)$ , about 30%.

*Proof.* Statement a) is equivalent to  $\beta_p(n) < \sqrt{n}$ . To verify this, observe that from (6.4) we deduce that  $\gamma_p(n) = 1$  if  $\sqrt{n} < p \leq n$ . In this range, if  $\beta_p(n) \geq \sqrt{n}$ , we would get that  $p\beta_p(n) > n$ , a contradiction with (6.5). Whenever  $\gamma_p(n) = \alpha \geq 2$ , we have that  $\beta_p(n) < p \leq n^{1/\alpha} \leq \sqrt{n}$ .

Claim b) is immediate.

c) Let us introduce  $J = J_n = \lfloor \sqrt{n} \rfloor - 1$  (so that  $J + 1 \leq \sqrt{n}$ ). For  $1 \leq j \leq J$ , primes  $p \leq n$  such that

$$\frac{n}{j+1} < p \leq \frac{n}{j}$$

satisfy  $\gamma_p(n) = 1$  and  $\beta_p(n) = j$ ; all other primes  $p \leq n$  not included among these  $J$  classes satisfy  $p \leq \frac{n}{J+1} \leq \sqrt{n}$ . Thus (6.10) follows.

For d), we write

$$\begin{aligned} \sum_{j=1}^{\infty} j \omega_j(n) &= \sum_{p \leq n} \beta_p(n) = \sum_{j=1}^J j \left[ \pi\left(\frac{n}{j}\right) - \pi\left(\frac{n}{j+1}\right) \right] + \sum_{p \leq \frac{n}{J+1}} \beta_p(n) \\ &= \sum_{j=1}^J \pi\left(\frac{n}{j}\right) - J \cdot \pi\left(\frac{n}{J+1}\right) + \sum_{p \leq \frac{n}{J+1}} \beta_p(n) = \sum_{j=1}^J \pi\left(\frac{n}{j}\right) + O\left(\frac{n}{\ln(n)}\right), \end{aligned}$$

where we have used summation by parts and the bounds

$$\sum_{p \leq n/(J+1)} \beta_p(n) \leq \frac{n}{J+1} \pi\left(\frac{n}{J+1}\right) = O\left(\frac{n}{\ln(n)}\right) \quad \text{and} \quad J \pi\left(\frac{n}{J+1}\right) = O\left(\frac{n}{\ln(n)}\right).$$

Now, from the error bound (6.7), we obtain

$$\sum_{j=1}^J \pi\left(\frac{n}{j}\right) = n \sum_{j=1}^J \frac{1}{j \ln(n/j)} + O\left(\frac{n}{\ln(n)}\right),$$

and, since,

$$\sum_{j=1}^J \frac{1}{j \ln(n/j)} = \underbrace{\int_1^{\sqrt{n}} \frac{1}{x \ln(n/x)} dx}_{=\ln(2)} + O\left(\frac{1}{\ln(n)}\right),$$

equation (6.11) follows.

Finally, (6.12) of statement e) follows by dividing (6.10) by  $\pi(n)$  and invoking the Prime Number Theorem.  $\square$

The following lemma furnishes some precise asymptotic estimate for the frequency  $\omega_1$ .

**Lemma 6.5.**

$$\omega_1(n) = \frac{1}{2} \frac{n}{\ln(n)} + O\left(\frac{n}{\ln^2(n)}\right)$$

*Proof.* The error bound (6.7) readily gives that

$$\pi(n) - \pi\left(\frac{n}{2}\right) = \frac{1}{2} \frac{n}{\ln(n)} + O\left(\frac{n}{\ln^2(n)}\right);$$

the result follows from (6.10) (for  $j = 1$ ) and the bound  $\pi(\sqrt{n}) = O\left(\frac{\sqrt{n}}{\ln(n)}\right)$ .  $\square$

We are now ready to estimate  $\mathbf{E}(\mathcal{T}_n)$ . We start by rewriting formula (6.6) as:

$$(6.13) \quad \mathbf{E}(\mathcal{T}_n) = n \int_0^\infty \left[ 1 - \prod_{j < \sqrt{n}} (1 - e^{-tj})^{\omega_j(n)} \right] dt.$$

By keeping just the factor corresponding to  $j = 1$  in (6.13), we obtain the following lower bound:

$$(6.14) \quad \mathbf{E}(\mathcal{T}_n) \geq n \int_0^\infty [1 - (1 - e^{-t})^{\omega_1(n)}] dt = n H_{\omega_1(n)},$$

where we have resorted to Lemma 2.3.

For an upper bound: using that  $e^{-jt} \leq e^{-2t}$  for  $j \geq 2$ , that  $\sum \omega_j(n) = \pi(n)$  and the identity  $1 - xy = 1 - x + x(1 - y)$ , we may bound

$$\begin{aligned} \mathbf{E}(\mathcal{T}_n) &\leq n \int_0^\infty [1 - (1 - e^{-t})^{\omega_1(n)} (1 - e^{-2t})^{\pi(n) - \omega_1(n)}] dt \\ &= n \int_0^\infty [1 - (1 - e^{-t})^{\omega_1(n)}] dt + n \int_0^\infty (1 - e^{-t})^{\omega_1(n)} [1 - (1 - e^{-2t})^{\pi(n) - \omega_1(n)}] dt \\ &= n H_{\omega_1(n)} + n \int_0^\infty (1 - e^{-t})^{\omega_1(n)} [1 - (1 - e^{-2t})^{\pi(n) - \omega_1(n)}] dt. \end{aligned}$$

Now, since  $1 - x^\delta \leq \delta(1 - x)$ , for  $0 \leq x \leq 1$  and  $\delta > 0$ , we may further bound

$$\begin{aligned} \mathbf{E}(\mathcal{T}_n) &\leq n H_{\omega_1(n)} + n(\pi(n) - \omega_1(n)) \int_0^\infty (1 - e^{-t})^{\omega_1(n)} e^{-2t} dt \\ &= n H_{\omega_1(n)} + n \frac{\pi(n) - \omega_1(n)}{(\omega_1(n) + 1)(\omega_1(n) + 2)} \leq n H_{\omega_1(n)} + n \frac{\pi(n) - \omega_1(n)}{\omega_1(n)^2} \end{aligned}$$

We have proved:

**Theorem 6.6.** *The mean value of  $\mathcal{T}_n$  satisfies:*

$$(6.15) \quad n H_{\omega_1(n)} \leq \mathbf{E}(\mathcal{T}_n) \leq n H_{\omega_1(n)} + n \frac{\pi(n) - \omega_1(n)}{\omega_1(n)^2}.$$

Finally,

**Corollary 6.7.**

$$\mathbf{E}(\mathcal{T}_n) = n \ln(n) - n \ln \ln(n) + n(\gamma - \ln(2)) + O(n/\ln(n)).$$

*Proof.* Using Lemma 2.3, the limit (6.12), for  $j = 1$ , and Lemma 6.5, we have that

$$\begin{aligned} H_{\omega_1(n)} &= \ln(\omega_1(n)) + \gamma + O\left(\frac{\ln(n)}{n}\right) \\ &= \ln\left(\frac{1}{2} \frac{n}{\ln(n)}\right) + \gamma + O\left(\frac{1}{\ln(n)}\right) + O\left(\frac{\ln(n)}{n}\right) = \ln\left(\frac{1}{2} \frac{n}{\ln(n)}\right) + \gamma + O\left(\frac{1}{\ln(n)}\right). \end{aligned}$$

Also, from the limit (6.12), for  $j = 1$ , we deduce

$$\frac{\pi(n) - \omega_1(n)}{\omega_1(n)^2} = O\left(\frac{\ln(n)}{n}\right).$$

Combining these two estimates, we get the result.  $\square$

Observe that, as a consequence of Corollary 6.7 and (6.8), we deduce that

$$(6.16) \quad \mathbf{E}(\tilde{\mathcal{T}}_n) - \mathbf{E}(\mathcal{T}) = n \ln(2) + O\left(\frac{n}{\ln(n)}\right).$$

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